CALCULATING THE INTEGRAL CHARACTERISTICS FOR A COMPRESSIBLE STREAM OF AN IDEAL GAS IN THE CASE OF ADIABATIC FLOW

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A simple calculation procedure is proposed to make it possible to use the results from experimental measurements of the velocity profile for a compressible stream to determine the integral characteristics of the boundary layer. This method is valid for the adiabatic flow of an ideal gas, provided that the experimental velocity profile can be satisfactorily approximated by an exponential function.

As is well known, the calculation of the integral characteristics for a dynamic boundary layer of a compressible stream on the basis of experimental data obtained in the measurement of the velocity profile is associated with laborious numerical integration. These calculations can be substantially simplified, given certain assumptions.

Let us consider the adiabatic flow of an ideal gas when Pr = 1. We will assume that the real distribution of the velocity can be satisfactorily approximated by an exponential function. The use of the exponential approximation in this case is purely theoretical and is not associated with a specific physical model of the flow governing the quantitative relationships of the process. At the same time, such an approximation exhibits significant advantages, since it permits us easily to obtain approximation parameters from the experimental points and also allows a significant simplification (as will be evident from the following) in the calculations, while retaining sufficient accuracy in the determination of the integral characteristics.

Let us write out the adopted assumptions:

$$\frac{u}{U_0} = (y/\delta)^{1/n},$$
$$\frac{1}{2}u^2 + c_pT = \frac{1}{2}u_0^2 + c_pT_0 = c_pT_{00}, \qquad (1)$$
$$\frac{p}{\rho} = RT.$$

In integrating the boundary-layer equations we frequently encounter an expression of the form

$$\int_0^\delta \frac{\rho}{\rho_0}\left(\frac{u}{u_0}\right)^i dy \qquad (i = 0,\ 1,\ 2,\ \ldots).$$

With (1) we can transform this integral

$$\frac{\rho}{\rho_0} = \frac{A-1}{A-(y/\delta)^{2/n}},\quad \left(\frac{u}{u_0}\right)^i = (y/\delta)^{i/n},$$

where

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TABLE 1. Values of the Integral J (n)

λ_0	n								
	2	3	4	5	6	7		9	10
0,30	0,005045	0,003788	0,003033	0,002528	0,002168	0,001898	0,001687	0,001519	0,001381
0,50	0,014247	0,010715	0,008590	0,007169	0,006152	0,005388	0,004793	0,004317	0,003926
0,70	0,028638	0,021601	0,017351	0,014503	0,012461	0,010924	0,009725	0,008764	0,007976
0,80	0,038020	0,028729	0,023107	0,019333	0,016624	0,014582	0,012989	0,011710	0,010662
0,90	0,049038	0,037132	0,029911	0,025055	0,021563	0,018929	0,016871	0,015218	0,013862
0,95	0,055215	0,041859	0,033747	0,028287	0,024357	0,021390	0,019071	0,017207	0,015676
0,98	0,059150	0,044875	0,036198	0,030354	0,026145	0,022967	0,020481	0,018483	0,016841
1,00	0,061872	0,046965	0,037898	0,031788	0,027386	0,024061	0,021460	0,019369	0,017650
1,02	0,064675	0,049119	0,039651	0,033268	0,028668	0,025192	0,022472	0,020285	0,018487
1,05	0,069038	0,052475	0,042386	0,035579	0,030670	0,026959	0,024054	0,021717	0,019796
1,07	0,072055	0,054799	0,044281	0,037181	0,032059	0,028186	0,025153	0,022712	0,020705
1,10	0,076748	0,058419	0,047236	0,039679	0,034228	0,030102	0,026870	0,024268	0,022127
1,20	0,093954	0,071743	0,058144	0,048931	0,042265	0,037212	0,033247	0,030051	0,027419
1,30	0,113860	0,087257	0,070903	0,059789	0,051726	0,045602	0,040787	0,036900	0,033694
1,70	0,232920	0,182149	0,150238	0,128164	0,111914	0,099417	0,089490	0,081402	0,074681
2,00	0,403743	0,323882	0,272324	0,235865	0,208528	0,187172	0,169974	0,155796	0,143886
2,20	0,625201	0,515470	0,442628	0,389905	0,349581	0,317533	0,291332	0,269439	0,250828

λ_0	n									
	11	12	13	14	15	16	17	18	19	20
0,30	0,001266	0,001169	0,001086	0,001013	0,000950	0,000894	0,000845	0,000800	0,000760	0,000724
0,50	0,003601	0,003325	0,003089	0,002884	0,002704	0,002546	0,002405	0,002279	0,002165	0,002063
0,70	0,007319	0,006761	0,006283	0,005868	0,005504	0,005183	0,004897	0,004641	0,004411	0,004202
0,80	0,009785	0,009043	0,008405	0,007851	0,007366	0,006937	0,006556	0,006214	0,005907	0,005628
0,90	0,012727	0,011764	0,010938	0,010220	0,009590	0,009034	0,008539	0,008095	0,007696	0,007333
0,95	0,014396	0,013310	0,012377	0,011566	0,010855	0,010227	0,009667	0,009166	0,008714	0,008304
0,98	0,015468	0,014303	0,013301	0,012431	0,011668	0,010993	0,010392	0,009854	0,009368	0,008929
1,00	0,016213	0,014993	0,013944	0,013032	0,012233	0,011526	0,010897	0,010332	0,009824	0,009363
1,02	0,016983	0,015706	0,014608	0,013654	0,012817	0,012078	0,011419	0,010828	0,010295	0,009813
1,05	0,018188	0,016822	0,015648	0,014628	0,013733	0,012941	0,012236	0,011604	0,011034	0,010517
1,07	0,019025	0,017599	0,016372	0,015305	0,014369	0,013542	0,012804	0,012143	0,011547	0,011007
1,10	0,020335	0,018813	0,017504	0,016365	0,015366	0,014482	0,013695	0,012989	0,012352	0,011775
1,20	0,025214	0,023338	0,021723	0,020318	0,019085	0,017993	0,017019	0,016146	0,015358	0,014644
1,30	0,031005	0,028715	0,026742	0,025024	0,023515	0,022178	0,020985	0,019915	0,018949	0,018072
1,70	0,069002	0,064137	0,059923	0,056235	0,052979	0,050083	0,047491	0,045156	0,043042	0,041118
2,00	0,133730	0,124956	0,117296	0,110545	0,104548	0,099183	0,094354	0,089982	0,086005	0,082371
2,20	0,234781	0,220782	0,208448	0,197488	0,187677	0,178837	0,170827	0,163531	0,156856	0,150723

TABLE 2. **Formulas for the Calculation of the Integral Characteristics of Plane-Parallel Flow, Using the Values of J(n)**

Displacement thickness $$\frac{\delta^*}{\delta}=\int_0^1\left(1-\frac{\rho u}{\rho_0 u_0}\right)d\left(\frac{y}{\delta}\right)$$	$1-n(A-1)J(n)$
Momentum loss thickness $$\frac{\delta^{**}}{\delta}=\int_0^1\frac{\rho u}{\rho_0 u_0}\left(1-\frac{u}{u_0}\right)d\left(\frac{y}{\delta}\right)$$	$n(A-1)[J(n)-J(n+1)]$
Energy loss thickness $$\frac{\delta^{***}}{\delta}=\int_0^1\frac{\rho u}{\rho_0 u_0}\left[1-\left(\frac{u}{u_0}\right)^2\right]d\left(\frac{y}{\delta}\right)$$	$n(A-1)[J(n)-J(n+2)]$
$$\frac{\delta^{*\prime}}{\delta}=\int_0^1\frac{\rho}{\rho_0}\left(1-\frac{u}{u_0}\right)d\left(\frac{y}{\delta}\right)$$	$n(A-1)[J(n-1)-J(n)]$
$$\eta(1)=\int_0^1\frac{\rho}{\rho_0}\,d\left(\frac{y}{\delta}\right)$$	$n(A-1)J(n-1)$

$$A=\frac{k+1}{k-1}\,\frac{1}{\lambda_0^2}\,.$$

Using the substitution $z=(y/\delta)^{1/n}$, we finally obtain

$$\int_0^\delta\frac{\rho}{\rho_0}\left(\frac{u}{u_0}\right)^i dy=\delta n(A-1)\int_0^1\frac{z^{n+i-1}}{A-z^2}\,dz=\delta n(A-1)\,J(n+i-1),\qquad(2)$$

where

$$J(n)=\int_0^1\frac{z^n}{A-z^2}\,dz.$$

The integral J(n) is easily calculated with the required degree of accuracy for any value of the exponent n (including a fractional exponent) by expanding the integrand in series and integrating term by term:

$$J(n)=\sum_{j=1}^{\infty}\frac{1}{2j+n-1}\,\frac{1}{A^j}\,.\qquad(3)$$

Series (3) converges more rapidly than a geometric progression, which always makes it possible to determine the number of terms in the series, which must be retained to achieve the specified accuracy. Calculation with (3) for most cases is considerably more economical than with the finite formulas which can be derived for whole values of n.

Such calculations were performed for the interval of values for n = 2-20 and λ_0 = 0.3-2.2 on the Ural-2 digital computer with an accuracy to 6 significant figures. This high accuracy can be explained by the fact that in determining the integral characteristics we have to work with the difference between close values of the integral J(n). The results of the calculations are shown in Table 1.*

It is easy to prove that under conditions (1) the calculation of the integral characteristics for the boundary layer of a compressible stream can be reduced to the calculations of the linear combination of earlier tabulated values of J(n). To carry out the calculations we have to find the exponent for the curve approximating the real velocity distribution and we must know the reduced velocity at the channel axis.

*The calculations in Table 1 have been carried out for k = 1.4. The results can easily be extended to any $(\lambda_0^2)_{\text{table}}=\lambda_0^2\,6(\bar{k}-1)/(\bar{k}+1)$.

TABLE 3. Formulas for the Calculation of the Integral Characteristics of Axisymmetric Flow, Using the Values of J(n)

Integral characteristic	Nonstabilized flow	Stabilized flow
Displacement thickness $\delta^* = \int_0^{r_0} \frac{r}{r_0}\left(1 - \frac{\rho u}{\rho_0 u_0}\right) dr$	$\delta n (A-1)\left[\frac{2r_0-\delta}{2nr_0(A-1)} - J(n) + \frac{\delta}{r_0} J(2n)\right]$	$r_0 n (A-1)\left[\frac{1}{2n(A-1)} - J(n) + J(2n)\right]$
Momentum loss thickness $\delta^{**} = \int_0^{r_0} \frac{r}{r_0} \frac{\rho u}{\rho_0 u_0}\left(1 - \frac{u}{u_0}\right) dr$	$\delta n (A-1)\left\{J(n) - J(n+1) - \frac{\delta}{r_0}[J(2n) - J(2n+1)]\right\}$	$r_0 n (A-1)[J(n) - J(n+1) - J(2n) + J(2n+1)]$
Energy loss thickness $\delta^{***} = \int_0^{r_0} \frac{r}{r_0} \frac{\rho u}{\rho_0 u_0}\left[1 - \left(\frac{u}{u_0}\right)^2\right] dr$	$\delta n (A-1)\left\{J(n) - J(n+2) - \frac{\delta}{r_0}[J(2n) - J(2n+2)]\right\}$	$r_0 n (A-1)[J(n) - J(n+2) - J(2n) + J(2n+2)]$
Fill factor $f = \frac{\int_0^{r_0} 2\pi r\rho u dr}{\pi r_0^2 \rho_0 u_0}$	$2n(A-1)\frac{\delta}{r_0}\left[J(n) - \frac{\delta}{r_0} J(2n)\right] + \left(1 - \frac{\delta}{r_0}\right)^2$	$2n(A-1)[J(n) - J(2n)]$
Average velocity $w = \frac{\int_0^{r_0} 2\pi r\rho u^2 dr}{\int_0^{r_0} 2\pi r\rho u dr}$	$u_0 \frac{(r_0-\delta)^2 + 2n(A-1)\delta[r_0 J(n+1) - \delta J(2n+1)]}{(r_0-\delta)^2 + 2n(A-1)\delta[r_0 J(n) - \delta J(2n)]}$	$u_0 \frac{J(n+1) - J(2n+1)}{J(n) - J(2n)}$

Table 2 shows the formulas for the determination of the integral characteristics of the boundary layer of a plane-parallel flow. Similar transformations are possible for axisymmetric flow. Here the velocity distribution is approximated by an exponential function of the form

$$\frac{u}{u_0} = \left(\frac{r_0 - r}{\delta}\right)^{1/n} \quad \text{(unstabilized flow)},$$

$$\frac{u}{u_0} = \left(\frac{r_0 - r}{r_0}\right)^{1/n} \quad \text{(stabilized flow)}.$$

The formulas for the axisymmetric flow are given in Table 3. For an incompressible fluid all of the relationships are reduced to those that are well known.

Let us expand the possibilities of the proposed method. We will examine the following integral:

$$\int_0^y \frac{\rho}{\rho_0}\left(\frac{u}{u_0}\right)^i dy = \delta n (A-1) \int_0^z \frac{z^{n+i-1}}{A - z^2}\, dz = \delta n (A-1) J_z(n+i-1),$$

where $z = (y/\delta)^{1/n}$ (by analogy with the previous),

$$J_z(n) = \int_0^z \frac{z^n}{A - z^2}\, dz.$$

Using the series expansion, we obtain

$$J_z(n) = \sum_{j=1}^{\infty} \frac{1}{2j+n+1} \frac{(y/\delta)^{\frac{2j+n-1}{n+1}}}{A^j}.$$

Then we can write

$$J_z(n) = J(n) \frac{\sum_{j=1}^{\infty} \frac{1}{2j+n-1} \frac{(y/\delta)^{\frac{2j+n-1}{n+1}}}{A^j}}{\sum_{j=1}^{\infty} \frac{1}{2j+n-1} \frac{1}{A^j}}. \tag{4}$$

To evaluate the magnitude of the fraction in (4), we remove the first terms of the infinite series in the numerator and in the denominator from the parantheses, and we replace the remaining terms with the geometric progressions

$$J_z(n) \approx J(n)(y/\delta) \frac{1 + \frac{n+1}{2} \sum_{j=1}^{\infty} \left[\frac{(y/\delta)^{\frac{2}{n+1}}}{A}\right]^{j-1}}{1 + \frac{n+1}{2} \sum_{j=1}^{\infty} \left(\frac{1}{A}\right)^{j-1}} = J(n)(y/\delta) \frac{1 + \frac{(n+1)A}{2[A - (y/\delta)^{2/n+1}]}}{1 + \frac{(n+1)A}{2(A-1)}}. \tag{5}$$

With formula (5) we can calculate the integral characteristics, although not for the entire thickness of the boundary layer, but at certain parts of it. In this case, it is sufficient to replace $J(n)$ by $J_z(n)$ in the expressions of Tables 2 and 3. This method makes it possible – applying the law of additivity – to calculate the integral characteristics even in the case in which the real profile is approximated by certain exponential relationships. Moreover, it can be used in converting to the Dorodnitsyn variables whose use enables us to reduce certain compressible-fluid problems to the problem of an incompressible fluid:

$$\eta(y/\delta) = \int_0^{y/\delta} \frac{\rho}{\rho_0}\, d(y/\delta) = n(A-1) J(n-1)(y/\delta) \frac{(A-1)\left[\frac{n+2}{2} A - (y/\delta)^{2/n}\right]}{[A - (y/\delta)^{2/n}]\left(\frac{n+2}{2} A - 1\right)}. \tag{6}$$

NOTATION

- y and r — are the instantaneous values of the transverse coordinate;
- r_0 — is the radius of the axisymmetric channel;
- δ — is the thickness of the boundary layer in the exponential approximation;
- n — is the parameter of the exponential function (see [1]);
- u — is the velocity;

γ	is the reduced velocity;
p, T, and ρ	are the semidynamic flow parameters;
R	is the gas constant;
c_p	is the specific heat capacity at constant pressure;
subscript 0	denotes the conditions at the axis or in the potential flow;
subscript 00	denotes the conditions in the case of isentropic stagnation.